Vettori aleatori

Si lancia 5 volte una moneta (p = prob. di ottenere "teste")

- X = numero di successi ~~nelle~~ nei 5 lanci
- Y = numero di successi nei primi 2 lanci

<u>Prob</u> di ottenere in totale 3 successi dei quali 2 nei primi 2 lanci

$$P(\{X=3\} \cap \{Y=2\}) = p(3,2)$$

$(\Omega, \mathcal{A}, P)$ fissato

Definizione Siano $X_1, X_2, \dots, X_m$ m v.a. discrete. Si chiama vettore aleatorio la funzione $X = (X_1, X_2, \dots, X_m)$ definita su Ω a valori in $\mathbb{R}^m$ da

$$\underline{X}(\omega) = \underbrace{(X_1(\omega), X_2(\omega), \dots, X_m(\omega))}$$

Es. $\Omega = \{0,1\}^5$ $\qquad (X_1, X_2)$

$\tilde{\omega} = (\underbrace{1, 0}, 1, 1, 1)$ $\qquad (X, Y, Z)$

$(X, Y) =$

$\omega \longmapsto (X(\omega), Y(\omega))$

$\tilde{\omega} \longmapsto (4, 1)$

$P(\underbrace{\{X=3\}}_{\in \mathcal{A}} \cap \underbrace{\{Y=2\}}_{\in \mathcal{A}})$

$P(X=3, Y=2) \longrightarrow \in \mathcal{A}$

$$X = (X_1, \ldots\ldots, X_m)$$

$$\{\underbrace{X_1 = x_1}, \underbrace{X_2 = x_2}, \ldots\ldots, \underbrace{X_m = x_m}\} \in \mathcal{A}$$

$$\forall \underline{(x_1, \ldots, x_m)} \in \mathbb{R}^m$$

$$x = (x_1, \ldots, x_m) \longmapsto P(X_1 = x_1, \ldots, X_m = x_m)$$

$$= P(X = x) = p(x)$$

$$f(x, y) = 3x + 5y$$

$$p : \mathbb{R}^m \longrightarrow \mathbb{R}$$

$p(x) = 0$ $\qquad$ $x = (x_1, \dots, x_m)$

$X = (X_1, \dots, X_m)$

$$P(X_1 = x_1, \dots, X_m = x_m) = 0$$

$\{X_1 = x_1\} = \emptyset$ $\qquad$ $x_1 \notin \operatorname{Im} X_1 = E_1$

Se esiste i tale che $x_i \notin \operatorname{Im} X_i = E_i$

allora $\{X_i = x_i\} = \emptyset$ e quindi

$$\{X_1 = x_1, \dots, X_m = x_m\} = \emptyset$$

$$\mathcal{E} = \boxed{\mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \mathcal{E}_m}$$

$$x = (x_1 \ldots x_i, x_{i+1} \ldots x_m)$$

$$x_i \downarrow \notin \mathcal{E}_i$$

<u>Definizione</u> La funzione $p : \mathbb{R}^m \to \mathbb{R}$ definita da $p(x) = P(X = x)$ si chiama densità <u>congiunta</u> di $(X_1, \ldots, X_m)$

Si chiama densità marginale i-esima la densità di X_i

$n = 2$

densità congiunta $(x, y) \longmapsto P(X = x, Y = y) = p(x,y)$

1^a densità marginale $x \longmapsto P(X = x) = p_X(x)$

2^a densità marginale $y \longmapsto P(Y = y) = p_Y(y)$

$x \longmapsto P(Y = x) = p_Y(x)$

$$(X, Y) \qquad p(x,y) = P(X=x, Y=y)$$

$$p_X(x) = P(X=x)$$

$$\{X=x\} = \{X=x\} \cap \Omega =$$

$$= \{X=x\} \cap \left(\bigcup_{y \in \mathbb{R} \cap \operatorname{Im} Y} \{Y=y\} \right)$$

$$= \bigcup_y \left(\{X=x\} \cap \{Y=y\} \right) =$$

$$\{X=x\} = \bigcup_{y \in \operatorname{Im} Y} \left(\underline{\{X=x,\ Y=y\}} \right)$$

$$P(X=x) = \sum_{y \in \cancel{\operatorname{Im} Y}\ \mathbb{R}} P(X=x,\ Y=y)$$

$$p_X(x) = \sum_{y \in \mathbb{R}} \underline{p(x,y)}$$

p

$$p_Y(y) = \sum_{x \in \mathbb{R}} p(x, y)$$

$$p_{X_1}(x_1) = \sum_{\substack{x_2 \in \mathbb{R} \\ x_3 \in \mathbb{R} \\ \vdots \\ x_m \in \mathbb{R}}} p(x_1, x_2, \dots, x_m)$$

$\vdots$

$$p_{X_2}(x_2) = \sum_{\substack{x_1 \\ x_3 \\ \vdots}} p(x_1, x_2, x_3 \dots x_m)$$

$\vdots$

$$(\underbrace{X, Y}, Z) \qquad p(x, y, z)$$

$$p_{X,Y}(x, y) = P(X = x, Y = y)$$

$$\underline{p_{X,Y}(x, y)} = \sum_{z \in \mathbb{R}} \underline{p(x, y, z)}$$

1) Un' urna contiene 5 palline numerate 1, 2, 3, 4, 5

Si eseguono 2 estrazioni con rimpiazzo

X = numero della 1^e pallina estratta

Y = numero della 2^e pallina estratta

Calcolare la densità congiunta di (X, Y) e le marginali

$$p(x,y) = P(X=x,\ Y=y)$$

$$(X,Y) : \Omega \longrightarrow \underline{\{1,2,3,4,5\}^2}$$

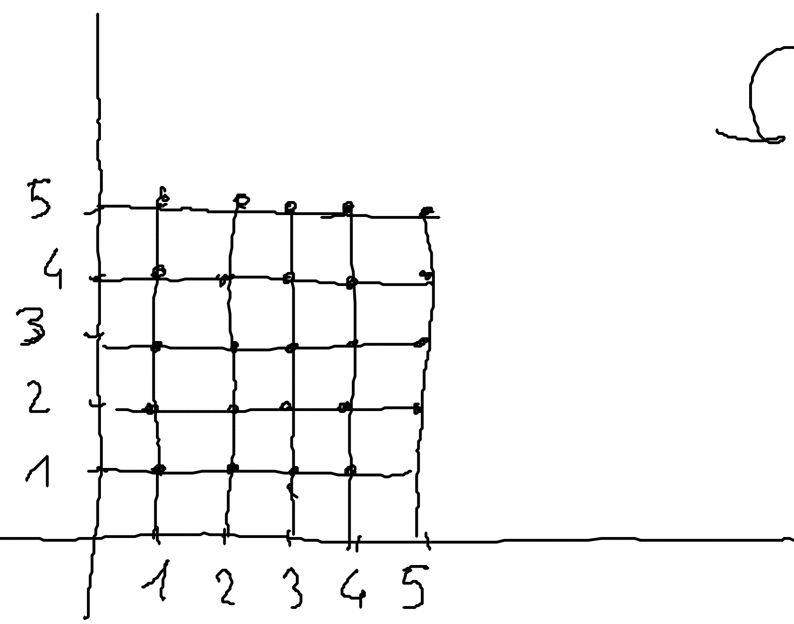
$$p(7,3) = P(X=7,\ Y=3) = 0$$

$$\Omega = \{1,2,3,4,5\}^2$$

$$P(\omega) = \frac{1}{25}$$

2^{η} επ

$$p(x,y) = \begin{cases} \frac{1}{25} & (x,y) \in \Omega \\ 0 & - \end{cases}$$



1^{η} επ

$$p(x,y) = \begin{cases} \frac{1}{25} & (x,y) \in \underline{\{1,2,3,4,5\}^2} \\ \underline{0} & \underline{\text{alnove}} \end{cases}$$

$$-\ p_X(x) = P(X = x) = \begin{cases} \frac{1}{5} & x \in \underline{\{1,2,3,4,5\}} \\ 0 & \text{\_\_\_\_} \end{cases}$$

$$-\ p_Y(y) = P(Y = y) = \begin{cases} \frac{1}{5} & y \in \underline{\{1,2,3,4,5\}} \\ 0 & \text{\_\_\_\_} \end{cases}$$

Schema prove indipendenti

$$\Omega = \{0,1\}^n$$

$$\omega = (\omega_1, \dots, \omega_n) \qquad \omega_i \in \{0,1\}$$

$$\boxed{X_i(\omega) = \omega_i}$$

$n=5 \quad \omega = (1, 1, 0, 1, 1)$

$$X_1(\omega) = 1 \qquad X_3(\omega) = 0$$
$$X_2(\omega) = 1 \qquad X_4(\omega) = 1$$
$$X_5(\omega) = 1$$

$n = 2 \qquad p$

$X_1 \qquad X_2$ sono indipendenti

$P(X_1 = 1, X_2 = 1) = P((1,1)) = p^2$

$P(X_1 = 1, X_2 = 0) = P((1,0)) = p(1-p)$

$P(X_1 = 0, X_2 = 1) = P((0,1)) = p(1-p)$

$P(X_1 = 0, X_2 = 0) = P((0,0)) = (1-p)^2$

$$P(X_1=1)P(X_2=1)=p^2$$

$$P(X_1=x, X_2=y) = P(X_1=x)P(X_2=y)$$

$X =$ n° di successi in n prove

$$X(\omega) = \sum_{i=1}^{n} \omega_i = \sum_{i=1}^{n} X_i(\omega)$$

$$\boxed{X = \sum_{i=1}^{n} X_i}$$

Una v. a. B (n, p) ha la stessa legge della somma di n v. a. Bernoulliane di parametro p e indipendenti

$$X = \sum_{i=1}^{n} X_i$$

Siano $X_1, X_2, \dots, X_n$

n v. a. discrete

Definizione Si dice che

$(X_1, \dots, X_n)$ sono v. a. indipendenti

se $\forall x_1, \forall x_2, \dots, \forall x_n$

$$P(X_1 = x_1, X_2 = x_2, \dots, X_n = x_n) =$$

$$= P(X_1 = x_1) P(X_2 = x_2) \cdot \dots \cdot P(X_n = x_n)$$

X, Y

Definizione Si dice che X e Y sono indipendenti se

$$P(X \in A, Y \in B) = P(X \in A)\,P(Y \in B)$$

$\forall A, B$ intervalli

__Definizione__ Si dice che

$X_1, \ldots\ldots, X_n$ sono v. a.

indipendenti se

$$P(X_1 \in A_1, X_2 \in A_2, \ldots, X_n \in A_n) =$$

$$= P(X_1 \in A_1) \cdot P(X_2 \in A_2) \cdot \ldots \cdot P(X_n \in A_n)$$

$\forall A_1, A_2, \ldots, A_n$ intervalli

Siano X, Y discrete

(1) $\left[\begin{array}{l} \forall x, y \\ \underline{P(X=x, Y=y)} = \underline{P(X=x)}\, \underline{P(Y=y)} \end{array} \right.$

(2) $\left[\begin{array}{l} \forall A, B \\ P(X \in A, Y \in B) = P(X \in A) P(Y \in B) \end{array} \right.$

Allora $(1) \iff (2)$

Dimostraz.

$(2) \Longrightarrow (1)$

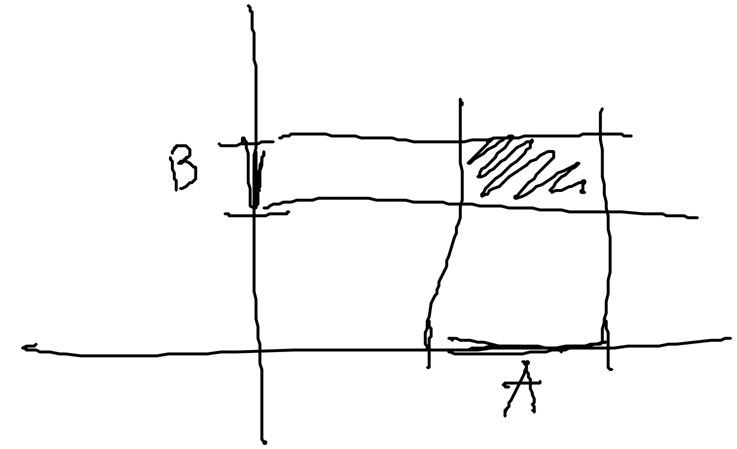
$A = \{x\}$ $\qquad B = \{y\}$

$P(X \in A, Y \in B) = P(X \in A)\, P(Y \in B)$

$P(X = x, Y = y) = P(X = x)\, P(Y = y)$

$(1) \Longrightarrow (2)$

$$P(X \in A, Y \in B) =$$



$$= P\big((X, Y) \in \underbrace{A \times B}_{S}\big) =$$

$$= \sum_{(x,y) \in \underbrace{A \times B}_{S}} p(x, y) = \sum_{\substack{x \in A \\ y \in B}} P(X = x, Y = y) =$$

$$= \sum_{x \in A,\, y \in B} P(X = x)\, P(Y = y)$$

$$=$$

$$= \sum_{x \in A} \sum_{y \in B} p_X(x) \cdot p_Y(y)$$

$$= \underbrace{\left(\sum_{x \in A} p_X(x) \right)} \left(\sum_{y \in B} p_Y(y) \right)$$

$$= P(X \in A) \cdot P(Y \in B)$$

$$U = \varphi(X)$$

$$V = \psi(Y)$$

$\boxed{\text{Prop}}$

Se X e Y sono indipendenti

allora U e V sono indipendenti

$X_1, X_2, \dots, X_n$ v. a. indipendenti

$(X_i)_{i \in I}$ $(X_i)_{i \in I^c}$

$X_1 \quad X_2 \quad X_3 \quad X_4 \quad X_5$

$I = \{2, 4\}$ $I^c = \{1, 3, 5\}$

(X_2, X_4) (X_1, X_3, X_5)

$$U = \varphi(\underbrace{X_2, X_4}) = X_2 + X_4$$

$$V = \psi(X_1, \underbrace{X_3, X_5}) = \frac{X_3 \cdot X_5}{X_1}$$

Se $X_1, X_2, X_3 \dots X_m$

sono indipendenti, allora

$$U = \varphi\left((X_i)_{i \in I}\right) \text{ e } V = \psi\left((X_i)_{i \in I^c}\right)$$

sono indipen.

$$p(\underbrace{x_1, \dots, x_m}) = p_{X_1}(x_1) \cdot p_{X_2}(x_2) \cdot \dots \cdot p_{X_m}(x_m)$$

prodotto tensoriale

$f(x) \qquad g(x)$

$$h(x) = f(x) \cdot g(x)$$

$p(x,y)$

$$p_X(x) = \sum_{y \in \mathbb{R}} p(x,y) \; ; \; p_Y(y) = \sum_{x \in \mathbb{R}} p(x,y)$$

$n = 2$

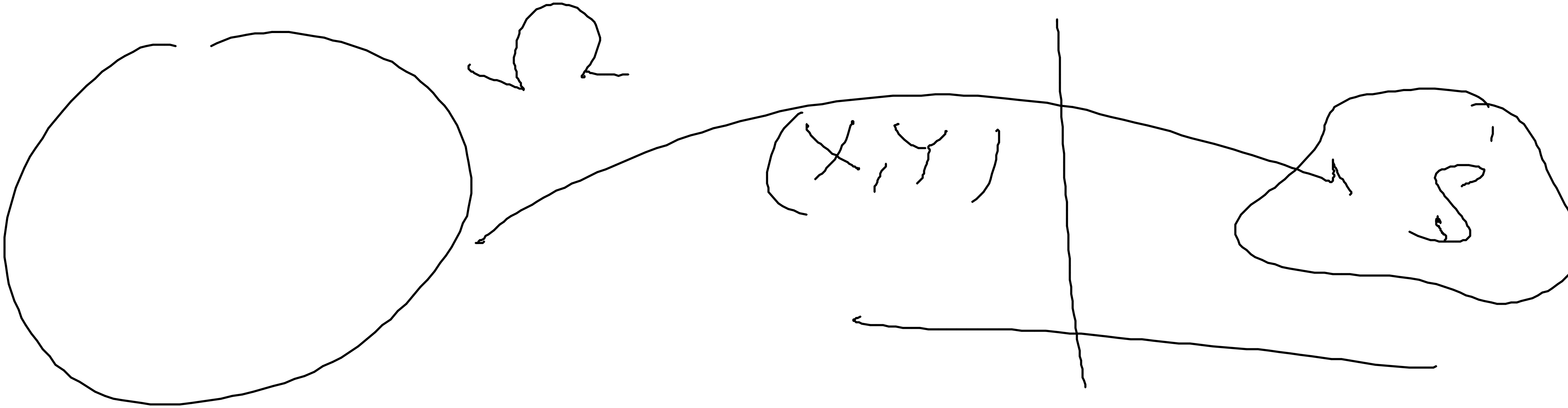
X_1

X_2

$$P(X \in I) = \sum_{x \in I} p(x)$$

caso unidimensionale

$$P(\underbrace{(X,Y)}_{\Omega} \in S)$$



$$\{(X,Y)\in S\} =$$

$$= \underbrace{\bigcup_{(x,y)\in S}}\ \underline{\underline{\{X=x,\ Y=y\}}}$$

$$P((X,Y)\in S) = \sum_{(x,y)\in S} \underbrace{P(X=x,\ Y=y)}_{p(x,y)}$$

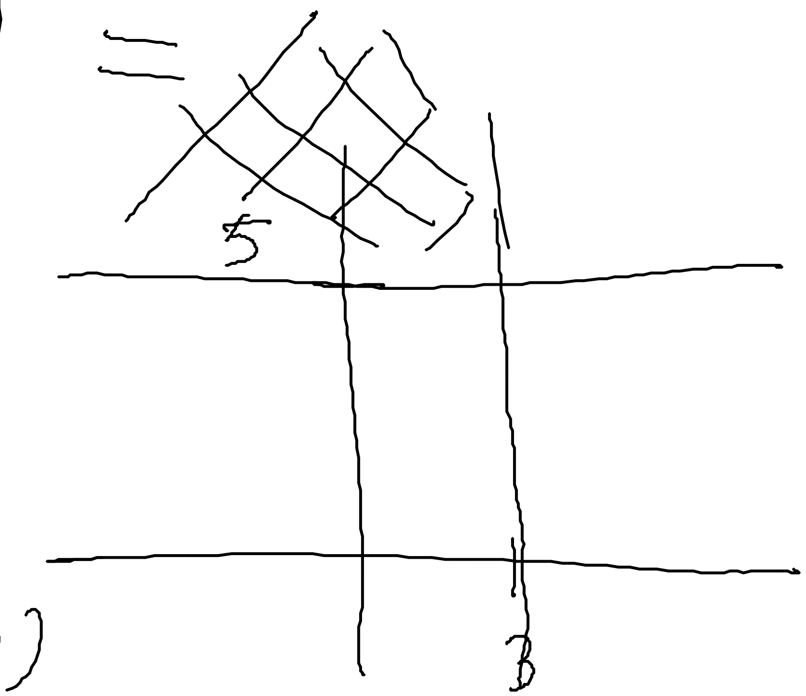
$$P(X \in S) = \sum_{x \in S} p(x)$$

$S \subset \mathbb{R}^m$

$X = (X_1, \dots, X_m)$

$x = (x_1, \dots, x_m)$

$$P(\underline{X \le 3, \ Y > 5}) =$$



$$= P((X,Y) \in ?)$$

$$= \sum_{(x,y) \in S} p(x,y) = \sum_{\substack{x \le 3 \\ y > 5}} p(x,y)$$

Variabili aleatorie indipendenti

$A \in \mathcal{A}$, $B \in \mathcal{A}$ sono indipendenti

se $P(A \cap B) = P(A)\,P(B)$

X, Y 2 v.a. discrete

$$\underset{A}{\underline{\{X = x\}}} \qquad \underset{B}{\underline{\{Y = y\}}} \qquad \forall x \ \forall y$$

Definizione. Si dice che X e Y sono 2 v. a. indipendenti se

$$\longrightarrow P(X=x, Y=y) = P(X=x)\,P(Y=y)$$

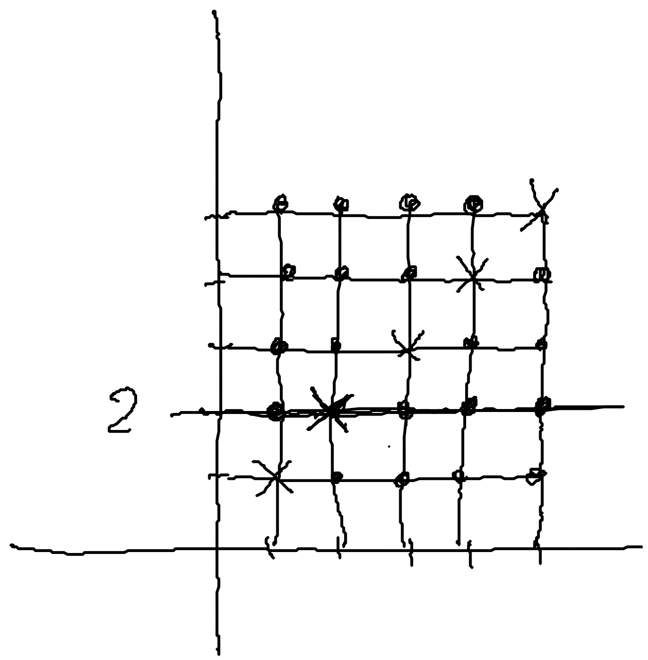
$$\forall x \in \mathbb{R}, \quad \forall y \in \mathbb{R}$$

Es. (caso 1) di prima)

$$p(x,y) = p_X(x) \cdot p_Y(y) \quad \underline{\underline{\forall x,\ \forall y}}$$

2) Ripetere, ma con estrazioni senza rimpiazzo

$$\Omega = \{1, 2, 3, 4, 5\}^2 \setminus \Delta$$



$$p(x,y) = \begin{cases} \frac{1}{20} & (x,y) \in \Omega \\ 0 & \text{—} \end{cases}$$

$$p_X(x) = \begin{cases} \frac{1}{5} & x \in \{1, 2, 3, 4, 5\} \\ 0 & \text{—} \end{cases}$$

$$P(Y=y) = \sum_{x} P(X=x, Y=y)$$

$$P(Y=2) = \sum_{x} \underline{\frac{1}{20}} = \frac{5}{20} = \frac{1}{4}$$

sbagliato!

$$P(Y=2) = \sum_{\substack{x=1 \\ x \neq 2}}^{5} \frac{1}{20} = \frac{4}{20} = \frac{1}{5}$$

$$\cancel{P_Y(y) = \begin{cases} 1 \end{cases}}$$

$$P_Y(y) = \begin{cases} \frac{1}{5} & y \in \{1,2,3,4,5\} \\ 0 & \end{cases}$$